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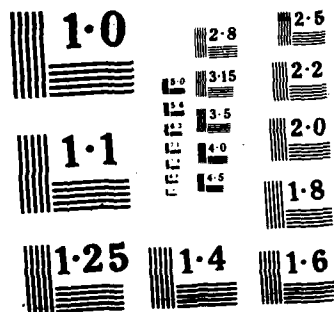
A FINITE ALGORITHM FOR THE LEAST TWO-NORM SOLUTION OF A 1/1  
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MRC Technical Summary Report #2945

A FINITE ALGORITHM FOR THE  
LEAST TWO-NORM SOLUTION OF A  
LINEAR PROGRAM

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July 1986

(Received April 22, 1986)

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A FINITE ALGORITHM FOR THE LEAST TWO-NORM SOLUTION  
OF A LINEAR PROGRAM

Stefano Lucidi<sup>\*</sup>

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ABSTRACT

By perturbing properly a linear program to a separable quadratic program it is possible to solve the latter in its dual variable space by iterative techniques such as sparsity-preserving SOR (successive overrelaxations). In this way large sparse linear programs can be handled.

In this paper we give a new computational criterion to check whether the solution of the perturbed quadratic programs provide the least 2-norm solution of the original linear program. This criterion improves on the criterion proposed in an earlier paper.

We also describe an algorithm for solving linear programs which is based on the SOR methods. The main property of this algorithm is that, under mild assumptions, it finds the least 2-norm solution of a linear program in a finite number of iterations.

AMS (MOS) Subject Classifications: 90C05, 90C20

Key Words: Linear programming, Quadratic programming

Work Unit Number 5 - Optimization and Large Scale Systems

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. DCR-8420963, and by the Italian National Council of Research (CNR).

# SIGNIFICANCE AND EXPLANATION

This paper provides the theory and practical algorithms for computing the unique smallest solution of a given linear program. The proposed method is capable of handling very large sparse linear programs that cannot be solved by the conventional simplex method.

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A FINITE ALGORITHM FOR THE LEAST TWO-NORM SOLUTION  
OF A LINEAR PROGRAM

Stefano Lucidi\*

1. INTRODUCTION

It was shown in [1] and [2] that the least 2-norm solution of a linear program can be computed by "properly" perturbing the linear program to a separable quadratic program and by solving the latter in its dual variable by iterative techniques such as SOR methods (see [3]). In this way large sparse linear programs, not solvable by standard pivotal methods can be handled (see [4], [1]). The main difficulty with this approach is that, until now, there is no easy way to know "a priori" if the perturbed quadratic problems is a "proper" perturbation of the original linear program.

In order to overcome this difficulty a computational criterion to check whether the solutions of the perturbed quadratic program was given in [5] and two algorithms where the perturbation parameter of the linear program is decreased during the computational procedure were proposed in [6] and [5].

We give now an outline of the paper. Section 2 contains the problem formulation. In Section 3 we characterize the least 2-norm solution of a linear program in terms of the constraints which are active at that point. This characterization will be utilized in subsequent sections of the paper. In Section 4 we turn our attention to the problem of the perturbation of a linear program and we propose a new criterion to check whether the solution of the perturbed problem is in fact the least 2-norm solution of the linear

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program. This criterion improves on that given in [5]. In the last section we describe an algorithm for solving linear programs which is based on the SOR method. The main property of this algorithm is that, under mild assumptions, it finds the least 2-norm solution of a linear program in a finite number of iterations.

We briefly describe the notation used. All the matrices and vectors are real. For the  $n \times n$  matrix  $A$  we denote row  $i$  by  $A_i$ , column  $j$  by  $A_{.j}$  and the element in row  $i$  and column  $j$  by  $A_{ij}$ . For  $x$  in the real  $n$ -dimensional Euclidean space  $R^n$ ,  $x_i$  denotes element  $i$  for  $i = 1, \dots, n$  and  $x_+$  denotes the vector with components  $(x_+)_i = \max\{x_i, 0\}$   $i = 1, \dots, n$ . All vectors are column vectors unless transposed by  $T$ . We will denote the 2-norm,  $(x^T x)^{1/2} = \left( \sum_{j=1}^n x_j^2 \right)^{1/2}$  by  $\|x\|$ , while  $R_+^n$  will denote the nonnegative orthant  $\{x : x \in R^n, x \geq 0\}$ . For a point  $c$  in  $R^n$  and a closed set  $X$  in  $R^n$  the 2-norm projection  $p_2(c, X)$  of the point  $c$  on  $X$  is defined

$$\|c - p_2(c, X)\| = \min_{x \in X} \|c - x\|.$$

For given index sets  $I, J$  we denote, as usual

$$|I| = \text{number of indices in } I$$

$$I \cap J = \{i : i \in I, i \in J\}$$

$$I \setminus J = \{i : i \in I, i \notin J\}$$

and  $0_{|I|}$  will be a null vector of dimension  $|I|$ . Finally we denote the identity matrix by  $E$ ,  $A_I$  will be the submatrix of  $A$  with rows  $A_i$   $i \in I$ ,  $A_{IJ}$  the submatrix of  $A$  with elements  $A_{ij}$ ,  $i \in I, j \in J$  and  $x_I$  will denote  $x_i$ ,  $i \in I$ .

## 2. PROBLEM FORMULATION

We consider the linear program

$$\begin{aligned} \text{Max } c^T x \\ \text{s.t. } Ax \leq b \\ x \geq 0 \end{aligned} \quad (1)$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $c \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$  and  $X = \{x : Ax \leq b, x \geq 0\} \neq \emptyset$ . Let  $\bar{X}$  denote the (possibly empty) optimal solution set of (1). The least 2-norm solution of (1),  $p_2(0, \bar{X})$ , can be characterized as the unique solution of the following problem

$$\begin{aligned} \text{Min } \frac{1}{2} \|x\|^2 \\ \text{s.t. } Ax \leq b \\ x \geq 0 \\ cx \geq \rho \end{aligned} \quad (2)$$

where  $\rho$  is the maximum value of (1).

Given any  $x \in X$ ,  $u \in \mathbb{R}^m$  and  $v \in \mathbb{R}^n$  we define the following index sets:

$$\begin{aligned} I_a(x) &:= \{i : A_i x = b_i\} \quad , \quad J_a(x) := \{i : x_i = 0\} \\ I_n(x) &:= \{i : A_i x < b_i\} \quad , \quad J_n(x) := \{i : x_i > 0\} \\ I_d(u) &:= \{i : u_i > 0\} \quad , \quad J_d(v) := \{i : v_i > 0\} \quad . \end{aligned}$$

Let  $I_b(x)$  and  $J_b(x)$  denote two index sets such that:

- i)  $I_b(x) \subseteq I_a(x)$ ,  $J_b(x) \subseteq J_a(x)$ .
- ii) The vectors  $A_i^T$ ,  $i \in I_b(x)$  and  $E_i^T$ ,  $i \in J_b(x)$  (where  $E_i$  is the  $i$ -th row of the identity matrix) are linearly independent.
- iii) All the vectors  $A_i^T$ ,  $i \in I_a(x) \setminus I_b(x)$  and  $E_i^T$ ,  $i \in J_a(x) \setminus J_b(x)$  are linear combinations of vectors  $A_i^T$ ,  $i \in I_b(x)$  and  $E_i^T$ ,  $i \in J_b(x)$ .



Furthermore we introduce the matrix  $B(x) \in R^{d \times d}$  and the vectors  $q(x)$ ,  $s(x) \in R^d$ , where  $d = |I_b(x)| + |J_b(x)|$ , as follows:

$$B(x) := \begin{bmatrix} E_{J_b}(x) \\ A_{I_b}(x) \end{bmatrix}, \quad q(x) := \begin{bmatrix} 0 \mid J_b(x) \mid \\ b_{I_b}(x) \end{bmatrix}, \quad s(x) := (B(x)B(x)^T)^{-1}q(x). \quad (3)$$

REMARK 1. From iii) it follows that:

$\forall i \in I_a(x) \setminus I_b(x)$  there exists a  $\tilde{z}_i \in R^d$  such that

$$A_i^T = B(x)^T \tilde{z}_i, \quad q(x)^T \tilde{z}_i = b_i$$

$\forall i \in J_a(x) \setminus J_b(x)$  there exists a  $\bar{z}_i \in R^d$  such that

$$E_i^T = B(x)^T \bar{z}_i, \quad q(x)^T \bar{z}_i = 0.$$

### 3. PROPERTIES OF THE LEAST 2-NORM SOLUTION OF A LINEAR PROGRAM

In this section we show some elementary but important properties for the least 2-norm solution of a linear program.

PROPOSITION 1. Let  $\bar{x}$  be a solution of the linear program (1) and  $(\bar{u}, \bar{v})$  be a solution of its dual. Then the following statements hold:

a) If  $\bar{x} = p_2(0, \bar{X})$  then  $\bar{x}$  is also the least 2-norm solution of the following system of equations

$$B(\bar{x})x = q(\bar{x}) \quad (4)$$

that is  $\bar{x} = B(\bar{x})^T (B(\bar{x})B(\bar{x})^T)^{-1} q(\bar{x}) = B(\bar{x})^T s(\bar{x})$ .

b) If  $\bar{x} = B(\bar{x})^T (B(\bar{x})B(\bar{x})^T)^{-1} q(\bar{x}) + Kc$ , for some  $K \in \mathbb{R}$ , and

$s(\bar{x})_{I_b(\bar{x}) \setminus I_d(\bar{u})} < 0$ ,  $s(\bar{x})_{J_b(\bar{x}) \setminus J_d(\bar{v})} > 0$  it follows that  $\bar{x} = p_2(0, \bar{X})$ .

c) Moreover, if the gradients of active constraints are linearly independent at  $\bar{x}$  then

$$\left\langle \begin{array}{l} \bar{x} = B(\bar{x})^T (B(\bar{x})B(\bar{x})^T)^{-1} q(\bar{x}) \\ s(\bar{x})_{I_b(\bar{x}) \setminus I_d(\bar{u})} < 0, s(\bar{x})_{J_b(\bar{x}) \setminus J_d(\bar{u})} > 0 \end{array} \right\rangle \iff \langle \bar{x} = p_2(0, \bar{X}) \rangle.$$

PROOF. First of all we set

$$I_a := I_a(\bar{x}), J_a := J_a(\bar{x}), I_n := I_n(\bar{x}), J_n := J_n(\bar{x}), I_d := I_d(\bar{u}), J_d := J_d(\bar{v}),$$

$$I_b := I_b(\bar{x}), J_b := J_b(\bar{x}), B := B(\bar{x}), q := q(\bar{x}), s := s(\bar{x}).$$

a) We prove this part by contradiction. Suppose that exists  $\tilde{x}$ , with  $\tilde{x} \neq \bar{x}$ , which is the least 2-norm solution of (4). Then all the points  $x(\theta) = \theta\tilde{x} + (1-\theta)\bar{x}$ ,  $\forall \theta \in \mathbb{R}$ , solve (4) and from the maximum principle it follows

$$\tilde{x}^T (x(\theta) - \tilde{x}) = 0. \quad (5)$$

By taking into account Remark 1 we have

$$\forall i \in J_a \setminus J_b \quad x_i(\theta) = \bar{z}_1^T Bx(\theta) = \bar{z}_1^T q = 0$$

$$\forall i \in I_a \setminus I_b \quad A_i x(\theta) = \tilde{z}^T Bx(\theta) = \tilde{z}^T q = b_i.$$

Hence  $x_{J_a}(\theta) = 0$ ,  $A_{I_a} x(\theta) = b_{I_a}$ ,  $\forall \theta \in R$ . Since  $\bar{x}_{J_n} > 0$  and  $A_{I_n} \bar{x}_{I_n} < b_{I_n}$  there exists a value  $\bar{\theta} \in (0, 1)$  such that  $x_{J_n}(\bar{\theta}) > 0$ ,  $A_{I_n} x(\bar{\theta}) < b_{I_n}$ . Therefore  $x(\bar{\theta}) \in X$  and it is, also, an optimal point for

(1): in fact from the KKT conditions for (1) it follows:

$$c^T x(\bar{\theta}) = c_{J_n}^T x(\bar{\theta})_{J_n} = \bar{u}_{I_a}^T A_{I_a J_n} x(\bar{\theta})_{J_n} = \bar{u}_{I_a}^T b_{I_a} = c^T \bar{x}.$$

Hence we can conclude that  $x(\bar{\theta})$  is feasible for problem (2). Now, by applying the maximum principle to problem (2) we obtain

$$\bar{x}^T (x(\bar{\theta}) - \bar{x}) > 0.$$

Furthermore by (5) we have

$$\begin{aligned} 0 < \bar{x}^T (x(\bar{\theta}) - \bar{x}) &= \bar{x}^T (\bar{\theta} \tilde{x} + (1-\bar{\theta})\bar{x} - \bar{x}) \\ &= \bar{\theta} (\tilde{x}^T \bar{x} - \|\bar{x}\|^2) = \bar{\theta} (\|\tilde{x}\|^2 - \|\bar{x}\|^2) < 0. \end{aligned}$$

Therefore  $\bar{x}$  is the least 2-norm solution of (3) that is (see [7])

$$\bar{x} = B^T (BB^T)^{-1} q = B^T s.$$

b) By assumption we have

$$\begin{aligned} \bar{x} = B^T s + Kc &= A_{I_b \cap I_d}^T s_{I_b \cap I_d} + A_{I_b \setminus I_d}^T s_{I_b \setminus I_d} + E_{J_b \cap J_d}^T s_{J_b \cap J_d} \\ &+ E_{J_b \setminus J_d}^T s_{J_b \setminus J_d} + Kc, \end{aligned} \quad (6)$$

and  $s_{I_b \setminus I_d} < 0$ ,  $s_{J_b \setminus J_d} > 0$ .

From the KKT condition for (1) it follows

$$\gamma(-c + A_{I_d}^T \bar{u}_{I_d} - E_{J_d}^T \bar{v}_{J_d}) = 0 \quad \forall \gamma \in R. \quad (7)$$

Now, by adding (6) and (7), we obtain

$$\begin{aligned} \bar{x} - (\gamma+K)c + A_{I_b \cap I_d}^T (\gamma \bar{u}_{I_b \cap I_d} - s_{I_b \cap I_d}) + A_{I_d \setminus I_b}^T \gamma \bar{u}_{I_d \setminus I_b} + A_{I_b \setminus I_d}^T (-s_{I_b \setminus I_d}) \\ - E_{J_b \cap J_d}^T (\gamma \bar{v}_{J_b \cap J_d} + s_{J_b \cap J_d}) - E_{J_d \setminus J_b}^T \gamma \bar{v}_{J_d \setminus J_b} - E_{J_b \setminus J_d}^T s_{J_b \setminus J_d} = 0. \end{aligned}$$

If we choose a  $\gamma > 0$  such that

$$\gamma+K > 0, \quad \gamma \bar{v}_{J_b \cap J_d} + s_{J_b \cap J_d} > 0, \quad \gamma \bar{u}_{I_b \cap I_d} - s_{I_b \cap I_d} > 0$$

and we set

$$\begin{aligned} u_{I_b \cap I_d}^* &:= \gamma \bar{u}_{I_b \cap I_d} - s_{I_b \cap I_d}, & u_{I_a \setminus I_b}^* &:= \gamma \bar{u}_{I_a \setminus I_b}, & u_{I_b \setminus I_d}^* &:= -s_{I_b \setminus I_d}, \\ v_{J_b \cap J_d}^* &:= \gamma \bar{v}_{J_b \cap J_d} + s_{J_b \cap J_d}, & v_{J_a \setminus J_b}^* &:= \gamma \bar{v}_{J_a \setminus J_b}, & v_{J_b \setminus J_d}^* &:= s_{J_b \setminus J_d}, \\ \gamma^* &= \gamma+K, & u_n^* &:= 0, & v_n^* &:= 0 \end{aligned}$$

we have that  $(\bar{x}, \gamma^*, u^*, v^*)$  satisfy the KKT conditions for (2).

c) ( $\Rightarrow$ ) It follows from part b)

( $\Leftarrow$ ) By the assumption that the gradients of the active constraints are linearly independent at  $\bar{x}$  we have

$$I_a = I_b, \quad J_a = J_b.$$

From part a) it follows

$$\bar{x} = B^T s = A_{I_d}^T s_{I_d} + A_{I_a \setminus I_d}^T s_{I_a \setminus I_d} + E_{J_d}^T s_{J_d} + E_{J_a \setminus J_d}^T s_{J_a \setminus J_d}. \quad (8)$$

Now  $\bar{x}$  solve also problem (2) and, hence, there exists  $u^* > 0, v^* > 0, \gamma^* > 0$

such that  $(\bar{x}, \gamma^*, u^*, v^*)$  satisfy the KKT conditions for (2), namely:

$$\bar{x} = A_{I_a}^T (-u_{I_a}^*) + E_{J_a}^T v_{J_a}^* + \gamma^* c$$

but by using the KKT condition for (1), that is

$$c = A_{I_d}^T \bar{u}_{I_d} - E_{J_d}^T \bar{v}_{J_d}$$

we obtain

$$\begin{aligned} \bar{x} = & A_{I_d}^T (\gamma \bar{u}_{I_d} - u_{I_d}^*) + A_{I_a \setminus I_d}^T (-u_{I_a \setminus I_d}^*) + E_{J_d}^T (v_{J_d}^* - \gamma \bar{v}_{J_d}) \\ & + E_{J_a \setminus J_d}^T v_{J_a \setminus J_d}^* . \end{aligned} \quad (9)$$

Then, by assumption, we have that the columns of  $B(\bar{x})^T$  are linearly independent and, hence, there must exist only one vector  $y$  such that:

$\bar{x} = B(\bar{x})^T y$ . Therefore, from (8) and (9) we obtain

$$s_{I_d} = \gamma \bar{u}_{I_d} - u_{I_d}^* , \quad s_{J_d} = v_{J_d}^* - \gamma \bar{v}_{J_d} ,$$

$$s_{I_b \setminus I_d} = s_{I_a \setminus I_d} = -u_{I_a \setminus I_d}^* < 0$$

$$s_{J_b \setminus J_d} = s_{J_a \setminus J_d} = v_{J_a \setminus J_d}^* > 0$$

and these last two inequalities complete the proof of part c).

We are now ready to establish a criterion for determining when the solution of the perturbed linear program is indeed the least 2-norm solution of the unperturbed linear program.

#### 4. PERTURBATION OF LINEAR PROGRAMS

In [1], [2] it was shown that the least 2-norm solution of the linear program (1) can be obtained by solving a "proper" quadratic perturbation of the linear program. In particular we have the following important result.

THEOREM 1. Let the linear program (1) feasible. Then

$$a) \quad i) \langle \bar{X} \neq \emptyset \rangle \implies \langle \exists \varepsilon^* > 0 : p_2(\frac{C}{\varepsilon}, X) = p_2(0, \bar{X}) \text{ for all } \varepsilon \in (0, \varepsilon^*) \rangle$$

$$ii) \left\langle \begin{array}{l} \bar{X} = 0 \\ x^* = p_2(0, \bar{X}) \end{array} \right\rangle \iff \langle \exists \varepsilon^* > 0, x^* : p_2(\frac{C}{\varepsilon}, X) = x^* \text{ for all } \varepsilon \in (0, \varepsilon^*) \rangle$$

$$b) \quad \sup_{x \in X} c^T x = \infty \iff \|p_2(\frac{C}{\varepsilon}, X)\| \rightarrow \infty \text{ as } \varepsilon \rightarrow 0^+.$$

PROOF. See [2].

We can note that  $p_2(\frac{C}{\varepsilon}, X)$  is also a solution of the problem

$$\begin{aligned} \text{Min } \frac{\varepsilon}{2} x^T x - c^T x \\ \text{s.t. } Ax \leq b \\ x \geq 0 \end{aligned} \tag{11}$$

and that the quadratic programming dual (see [8]) to (11) is

$$\begin{aligned} \text{Min } \frac{1}{2} \|A^T u - v - c\|^2 + \varepsilon b^T u \\ \text{s.t. } (u, v) \geq 0 \end{aligned} \tag{12}$$

where the primal and dual variable  $x$  and  $(u, v)$  are related by

$$x = \frac{1}{\varepsilon} (-A^T u + v + c). \tag{13}$$

Until now there was no simple way of determining "a priori" the parameter  $\varepsilon^*$  of Theorem 1. The next result which gives on "a posteriori" computational scheme for  $\varepsilon^*$  shows why it is difficult to compute  $\varepsilon^*$  "a priori". In fact

we can note that  $\epsilon^*$  depends strictly on the least 2-norm solution of (1) (by means of the vector  $s(\bar{x})$ , defined by (3)) and on the optimal solution of the dual of the linear program (1).

PROPOSITION 2. Assume  $\bar{x} \neq \beta$ . Let the gradients of the active constraints be linearly independent at  $\bar{x} = p_2(0, \bar{x})$  and let  $(u^*, v^*)$  be the solution point of the dual of the linear program (1). Then it follows that

$$p_2\left(\frac{c}{\epsilon}, \bar{x}\right) = \bar{x} = p_2(0, \bar{x}) \quad \forall \epsilon \in (0, \epsilon^*)$$

where  $\epsilon^*$  is given by

$$\epsilon^* = \min \left\{ \left[ \min_{i \in \{i \in I_d(u^*), s(\bar{x})_i < 0\}} u_i^* / s(\bar{x})_i \right], \left[ \min_{i \in \{i \in J_d(v^*), s(\bar{x})_i < 0\}} -v_i^* / s(\bar{x})_i \right] \right\} \quad (14)$$

where  $s(\bar{x})$  is defined by (3) and  $I_d(u^*) = \{i : u_i^* > 0\}$  and  $J_d(v^*) = \{i : v_i^* > 0\}$ .

PROOF. We recall the definitions given in Section 2 and we set  $I_a := I_a(\bar{x})$ ,  $J_a := J_a(\bar{x})$ ,  $I_n := I_n(\bar{x})$ ,  $J_n := J_n(\bar{x})$ ,  $I_d := I_d(u^*)$ ,  $J_d := J_d(v^*)$ ,  $s := s(\bar{x})$ .

Now, from the KKT conditions for (1) we have

$$-c + A_{I_d}^T u_{I_d}^* - E_{J_d}^T v_{J_d}^* = 0 \quad (15)$$

By using part c) of Proposition 1 we obtain

$$\epsilon(\bar{x} - A_{I_d}^T s_{I_d} - A_{I_a \setminus I_d}^T s_{I_a \setminus I_d} - E_{J_d}^T s_{J_d} - E_{J_a \setminus J_d}^T s_{J_a \setminus J_d}) = 0 \quad (16)$$

where  $s_{I_a \setminus I_d} < 0$  and  $s_{J_a \setminus J_d} > 0$ .

Hence, by adding (15) and (16), we obtain

$$\epsilon \bar{x} - c + A_{I_d}^T (u_{I_d}^* - \epsilon s_{I_d}) + A_{I_a \setminus I_d}^T (-\epsilon s_{I_a \setminus I_d}) - E_{J_d}^T (v_{J_d}^* + \epsilon s_{J_d}) - E_{J_a \setminus J_d}^T (\epsilon s_{J_a \setminus J_d}) = 0.$$

If we choose  $\varepsilon^*$  as the largest value of  $\varepsilon$  such that

$$u_{I_d}^* - \varepsilon s_{I_d} > 0, \quad v_{J_d}^* + \varepsilon s_{J_d} > 0$$

(that is if  $\varepsilon^*$  is given by (14)) we can define  $\forall \varepsilon \in (0, \varepsilon^*]$

$$u(\varepsilon)_{I_d} = u_{I_d}^* - \varepsilon s_{I_d}, \quad u(\varepsilon)_{I_a \setminus I_d} := -\varepsilon s_{I_a \setminus I_d}, \quad u(\varepsilon)_{I_n} := 0, \quad (17)$$

$$v(\varepsilon)_{J_d} := v_{J_d}^* + \varepsilon s_{J_d}, \quad v(\varepsilon)_{J_a \setminus J_d} := \varepsilon s_{J_a \setminus J_d}, \quad v(\varepsilon)_{J_n} := 0, \quad (18)$$

Then it turns out that  $(\bar{x}, u(\varepsilon), v(\varepsilon))$  is the unique solution of the KKT conditions for problem (11),  $\forall \varepsilon \in (0, \varepsilon^*]$ .

Therefore we have that

$$\bar{x} = p_2(0, \bar{X}) = p_2\left(\frac{C}{\varepsilon}, X\right) \quad \forall \varepsilon \in (0, \varepsilon^*]$$

and that  $(\bar{u}(\varepsilon), \bar{v}(\varepsilon))$  is the unique optimal solution of the dual problem (12).

In order to overcome the inherent difficulty of determining  $\varepsilon^*$ , "a priori" a practical computational criterion was proposed in [5] to check whether the solutions of the perturbed quadratic programs provide the least 2-norm solution of the original linear program. This criterion required two points  $(x_i, u_i, v_i)$ ,  $i = 1, 2$ , that satisfy the KKT conditions for problem (11) with two different values for  $\varepsilon$ .

In the sequel we give an improved new criterion that requires only one point  $(x, u, v)$  which satisfies the KKT conditions for problem (11). However this new criterion requires a matrix inversion. By contrast, no matrix inversion was required in [5].

In the next theorem we make use of the index sets  $I_a(x)$ ,  $J_a(x)$ ,  $I_n(x)$ ,  $J_n(x)$ ,  $I_b(x)$ ,  $J_b(x)$ ,  $I_d(u)$ ,  $J_d(v)$ , the matrix  $B(x)$ , and the vectors  $q(x)$  and  $s(x)$  which were defined in section 2. Furthermore we assume that



$I_a(\bar{x}) \cup J_a(\bar{x}) \neq \emptyset$  where  $\bar{x}$  is the solution of (11), because otherwise we have trivially that  $\bar{x} \neq p_2(0, \bar{X})$ .

THEOREM 2. Let  $\bar{x}$  be a solution of problem (11), that is  $\bar{x} = p_2(\frac{c}{\epsilon}, X)$ , let  $(\bar{u}, \bar{v})$  be a solution of the dual problem (12) and let  $\epsilon^*$  be the optimal value for  $\epsilon$  given by (14). Then

$$a) \left\langle \begin{array}{l} \bar{u}_{I_b(\bar{x})} > -\epsilon s(\bar{x})_{I_b(\bar{x})}, \bar{v}_{J_b(\bar{x})} > \epsilon s(\bar{x})_{J_b(\bar{x})} \\ [E - B^T(\bar{x})(B(\bar{x})B(\bar{x})^T)^{-1}B(\bar{x})]c = \gamma c \\ \text{for some } \gamma < 1 \end{array} \right\rangle \implies \left\langle \begin{array}{l} \bar{X} \neq \emptyset \\ \bar{x} = p_2(\frac{c}{\epsilon}, X) = p_2(0, \bar{X}) \end{array} \right\rangle$$

and  $(u^*, v^*)$  are optimal for the dual of the linear program (1) where

$$u^*_{I_b(\bar{x})} := \frac{(\bar{u}_{I_b(\bar{x})} + \epsilon s(\bar{x})_{I_b(\bar{x})})}{1 - \gamma}, u^*_{I_a(\bar{x}) \setminus I_b(\bar{x})} := \frac{\bar{u}_{I_a(\bar{x}) \setminus I_b(\bar{x})}}{1 - \gamma}, u^*_{I_n(\bar{x})} := 0 \quad (19)$$

$$v^*_{J_b(\bar{x})} := \frac{(\bar{v}_{J_b(\bar{x})} - \epsilon s(\bar{x})_{J_b(\bar{x})})}{1 - \gamma}, v^*_{J_a(\bar{x}) \setminus J_b(\bar{x})} := \frac{\bar{v}_{J_a(\bar{x}) \setminus J_b(\bar{x})}}{1 - \gamma}, v^*_{J_n(\bar{x})} := 0. \quad (20)$$

b) If the gradients of the active constraints are linearly independent at  $\bar{x}$ , then it follows that:

$$\left\langle \begin{array}{l} \bar{u}_{I_a(\bar{x})} > -\epsilon s(\bar{x})_{I_a(\bar{x})}, \bar{v}_{J_a(\bar{x})} > \epsilon s(\bar{x})_{J_a(\bar{x})} \\ [E - B^T(\bar{x})(B(\bar{x})B(\bar{x})^T)^{-1}B(\bar{x})]c = 0 \end{array} \right\rangle \iff \left\langle \begin{array}{l} \bar{X} \neq \emptyset \\ 0 < \epsilon < \epsilon^* \\ \bar{x} = p_2(\frac{c}{\epsilon}, X) = p_2(0, \bar{X}) \end{array} \right\rangle$$

and, if  $\bar{x} = p_2(0, \bar{X})$ , the optimal solution  $(u^*, v^*)$  of the dual of the linear program (1) is given by (19) and (20) with  $\gamma = 0$ ,  $I_a(\bar{x}) = I_b(\bar{x})$  and  $J_a(\bar{x}) = J_b(\bar{x})$ .

c) If the gradients of the active constraints are linearly independent at  $\bar{x}$  and the strict complementarity holds for the linear program (1) at the least 2-norm solution  $p_2(0, \bar{X})$ , then it follows that:

$$\left\langle \begin{array}{l} \bar{u} > -\epsilon s(\bar{x}), \quad \bar{v} > \epsilon s(\bar{x}) \\ I_a(\bar{x}) > -\epsilon s(\bar{x}), \quad I_a(\bar{x}) > \epsilon s(\bar{x}) \\ J_a(\bar{x}) > -\epsilon s(\bar{x}), \quad J_a(\bar{x}) > \epsilon s(\bar{x}) \\ [E - B^T(\bar{x})(B(\bar{x})B(\bar{x})^T)^{-1}B(\bar{x})]c = 0 \end{array} \right\rangle \Leftrightarrow \left\langle \begin{array}{l} \bar{X} \neq \emptyset \\ 0 < \epsilon < \epsilon^* \\ \bar{x} = p_2\left(\frac{c}{\epsilon}, X\right) = p_2(0, \bar{X}) \end{array} \right\rangle$$

PROOF. Again we set  $I_a := I_a(\bar{x})$ ,  $J_a := J_a(\bar{x})$ ,  $I_n := I_n(\bar{x})$ ,  $J_n := J_n(\bar{x})$ ,  $I_b := I_b(\bar{x})$ ,  $J_b := J_b(\bar{x})$ ,  $I_d := I_d(\bar{u})$ ,  $J_d := J_d(\bar{v})$ ,  $B := B(\bar{x})$ ,  $q := q(\bar{x})$ ,  $s := s(\bar{x})$ .

a) Define

$$\bar{y} := \epsilon \bar{x} - c.$$

Since the point  $(\bar{x}, \bar{u}, \bar{v})$  satisfies the KKT condition for problem (11), the point  $(\bar{y}, \bar{u}, \bar{v})$  satisfies the KKT conditions of the following problem

$$\begin{aligned} \text{Min } \frac{1}{2} \|\bar{y}\|^2 \\ A\bar{y} &\leq \epsilon b - Ac \\ \bar{y} &> -c. \end{aligned} \tag{21}$$

Therefore  $\bar{y}$  is the optimal solution of (21).

By repeating the same steps of the proof of part a) of Proposition 1 we have that  $\bar{y}$  is the least 2-norm solution of the following system of equations

$$By = \varepsilon q - Bc \quad . \quad (22)$$

In fact, suppose not. Then there exists a  $\tilde{y}$ ,  $\tilde{y} \neq \bar{y}$ , which is the least norm solution of (22). Then  $\forall \theta$  the points  $y(\theta) = \theta \tilde{y} + (1-\theta)\bar{y}$  solve (22) and (recalling Remark 1)

$$\forall i \in J_a \setminus J_b \quad y_i(\theta) = \tilde{z}_i^T B y(\theta) = \tilde{z}_i^T (\varepsilon q - Bc) = -c_i$$

$$\forall i \in I_a \setminus I_b \quad A_i y(\theta) = \tilde{z}_i^T B y(\theta) = \tilde{z}_i^T (\varepsilon q - Bc) = \varepsilon b_i - A_i c \quad .$$

Moreover, since  $\bar{y}_{J_n} > -c_{J_n}$  and  $A_{I_n} \bar{y} < \varepsilon b_{I_n} - A_{I_n} c$ , there exists a value  $\theta \in (0,1)$  such that  $y(\theta)_{J_n} > -c_{J_n}$  and  $A_{I_n} y(\theta) < \varepsilon b_{I_n} - A_{I_n} c$ . Therefore  $y(\theta) = \{y' : Ay' < \varepsilon b - Ac, y' > -c\}$ .

Now, by applying again the maximum principle, we obtain

$$0 < \bar{y}^T (y(\theta) - \bar{y}) = \bar{\theta} (\bar{y}^T \tilde{y} - \|\bar{y}\|^2) = \bar{\theta} (\|\tilde{y}\|^2 - \|\bar{y}\|^2) < 0 \quad .$$

Hence we can conclude that  $\bar{y}$  is the least norm solution of (22), namely

$$\bar{y} = B^T (BB^T)^{-1} [\varepsilon q - Bc]$$

from which

$$\begin{aligned} \varepsilon \bar{x} &= [E - B^T (BB^T)^{-1} B] c + \varepsilon B^T (BB^T)^{-1} q \\ &= [E - B^T (BB^T)^{-1} B] c + \varepsilon B^T s \quad . \end{aligned}$$

Now by recalling the KKT conditions for (11) we have

$$\varepsilon \bar{x} - c + A_{I_a}^T \bar{u}_{I_a} - E_{J_a}^T \bar{v}_{J_a} = 0$$

from which we obtain

$$[E - B^T (BB^T)^{-1} B] c + \varepsilon B^T s - c + A_{I_a}^T \bar{u}_{I_a} - E_{J_a}^T \bar{v}_{J_a} = 0$$

$$[E - B^T (BB^T)^{-1} B] c - c + A_{I_b}^T (\bar{u}_{I_b} + \varepsilon s_{I_b}) + A_{I_a \setminus I_b}^T \bar{u}_{I_a \setminus I_b}$$

$$- E_{J_b}^T (\bar{v}_{J_b} - \varepsilon s_{J_b}) - E_{J_a \setminus J_b}^T \bar{v}_{J_a \setminus J_b} = 0 \quad .$$

By assumptions we have  $[E - B^T(BB^T)^{-1}B] = \gamma c$  with some  $\gamma < 1$

$$\bar{u}_{I_b} + \varepsilon s_{I_b} > 0, \quad \bar{v}_{J_b} - \varepsilon s_{J_b} > 0.$$

Therefore if we set

$$u_{J_b}^* := \frac{\bar{u}_{I_b} + \varepsilon s_{I_b}}{1 - \gamma}, \quad u_{I_a \setminus I_b}^* := \frac{\bar{u}_{I_a \setminus I_b}}{1 - \gamma}, \quad u_{I_n}^* := 0,$$

$$v_{J_b}^* := \frac{\bar{v}_{J_b} - \varepsilon s_{J_b}}{1 - \gamma}, \quad v_{J_a \setminus J_b}^* := \frac{\bar{v}_{J_a \setminus J_b}}{1 - \gamma}, \quad v_{J_n}^* := 0,$$

we have that  $(\bar{x}, u^*, v^*)$  satisfy the KKT conditions for the linear program (1).

b) ( $\Rightarrow$ ) The proof of this implication follows directly from part a).

( $\Leftarrow$ ) By using the KKT conditions for linear program (1) we know that there exist  $(\tilde{u}, \tilde{v})$  such that

$$-c + A_{I_a}^T \tilde{u}_{I_a} - E_{J_a}^T \tilde{v}_{J_a} = 0$$

from which

$$c = B^T \begin{pmatrix} \tilde{v}_{J_a} \\ \tilde{u}_{I_a} \end{pmatrix}.$$

Therefore we have

$$[E - B^T(BB^T)^{-1}B]c = [E - B^T(BB^T)^{-1}B]B^T \begin{pmatrix} \tilde{v}_{J_a} \\ \tilde{u}_{I_a} \end{pmatrix} = 0.$$

Now, by repeating the same steps of proof of Proposition 2 and by using (17)

and (18) we obtain that

$$\bar{u}_{I_a} > -\varepsilon s_{I_a}, \quad \bar{v}_{J_a} > \varepsilon s_{J_a}.$$

c) The proof of part c) follows from the proof of part b) and the proof of Proposition 2 (see (17) and (18)).

REMARK 2. Part b) of Theorem 2 characterizes the solvability of a linear program in terms of the solvability of a convex quadratic function minimization on the non negative orthant (problem (12)) without any "a priori" assumptions regarding the solvability of the linear program (1).

We now turn our attention to computational procedures for determining the least 2-norm solution of a linear program.

## 5. AN ALGORITHM WITH FINITE TERMINATION FOR LINEAR PROGRAMMING

The quadratic programming problem (12) can be solved by using a sparsity-preserving SOR algorithm introduced in [4]. More specifically we have the following algorithm where we have assumed that  $A_j \neq 0$ ,  $\forall j = 1, \dots, m$ .

### LPSOR Algorithm

Choose  $(u^0, v^0) \in R_+^{m+n}$ ,  $\omega \in (0, 2)$  and  $\epsilon > 0$ .

Having  $(u^k, v^k)$  compute  $(u^{k+1}, v^{k+1})$  as follows:

$$u_j^{k+1} = \left( u_j^k - \frac{\omega}{\|A_j\|^2} \left( A_j \left( \sum_{\ell=1}^{j-1} (A^\top)_{\ell} u_\ell^{k+1} + \sum_{\ell=j}^m (A^\top)_{\ell} u_\ell^k - v^k - c \right) + \epsilon b_j \right) \right)_+,$$

$$v^{k+1} = (v^k - \omega(-A^\top u^{k+1} + v^k + c))_+.$$

$k = 1, \dots, m$

The principal and computationally-distinguishing features of this SOR algorithm are that it preserves the sparsity structure of the problem and require only simple operations, and, hence, very large problems can be tackled.

We refer to [2] and [4] for a more complete description of sparsity-preserving SOR algorithm, here we only recall the following convergence result for the LPSOR Algorithm

**PROPOSITION 3.** Assume that  $\bar{x} \neq \emptyset$  and that the gradients of the active constraints of the linear program (1) at the optimal point  $\bar{x} = p_2(0, \bar{x})$  are linearly independent. Let  $\{(u^k, v^k)\}$  be the sequence generated by the LPSOR Algorithm. Then

a) There exists a real positive number  $\epsilon^*$  such that for each  $\epsilon \in (0, \epsilon^*)$ , the sequence  $\{(u^k, v^k)\}$  converges to a point  $(u(\epsilon), v(\epsilon))$  which solves problem (12) and the corresponding  $x(\epsilon)$  determined by (13) is independent of

$\varepsilon$  and  $x(\varepsilon) = \bar{x} = p_2(0, \bar{X})$ .

b) Moreover, if strict complementarity holds for problem (1) at the least 2-norm solution  $\bar{x} = p_2(0, \bar{X})$ , then there exists a real positive number  $\varepsilon^{**}$ ,  $\varepsilon^{**} < \varepsilon^*$ , such that for each  $\varepsilon \in (0, \varepsilon^{**}]$  there exists a  $k^*$  such that for all  $k > k^*$  the following hold:

$$\{i : u_i^k > 0\} = \{i : u_i^* > 0\} = I_a(\bar{x})$$

$$\{i : u_i^k = 0\} = \{i : u_i^* = 0\} = I_n(\bar{x})$$

$$\{i : v_i^k > 0\} = \{i : v_i^* > 0\} = J_a(\bar{x})$$

$$\{i : v_i^k = 0\} = \{i : v_i^* = 0\} = J_n(\bar{x})$$

where  $(u^*, v^*)$  is the optimal solution of the dual of the linear program (1).

( $I_a(x), I_n(x), J_a(x), J_n(x)$  were defined in section 2.)

PROOF. See [5].

In applying part 1) of Proposition 2 we must be able to select a value of  $\varepsilon$  such that  $\varepsilon < \varepsilon^*$ . In order to ensure that  $\varepsilon < \varepsilon^*$  we may have to choose very small values for  $\varepsilon$  yield a very slow convergence of the LPSOR Algorithm when applied to the problem (12).

In order to overcome both the lack of a practical "a priori" selection procedure for the parameter  $\varepsilon^*$  and the need to solve problem (12) exactly many times for a decreasing sequence of  $\varepsilon$  values (until one of the criteria of Theorem 2, Corollary 1 or Theorem 2 of [5] is satisfied) we propose an algorithm which is based on the results of the preceding sections

The key idea of this algorithm is similar to that of the algorithms proposed in [5], [6], where problem (12) is solved only approximately after which  $\varepsilon$  is decreased in a prescribed manner.

The main distinguishing property of the present algorithm is that, under mild assumptions, it finds the least 2-norm solution of the linear program (1) in a finite number of iterations. The algorithm described below makes use of two preselected sequences  $\{\epsilon^k\}$  and  $\{N^k\}$  such that

$$\lim_{k \rightarrow \infty} \epsilon^k = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} N^k = +\infty. \quad (23)$$

#### ALGORITHM I

Choose  $(u^0, v^0) \in \mathbb{R}_+^{m+n}$ . Having  $(u^{k-1}, v^{k-1})$ ,  $(u^k, v^k)$  is obtained by applying  $N^k$  iterations of the SOR algorithm to the problem (12) with  $\epsilon = \epsilon^k$  and by using  $(u^{k-1}, v^{k-1})$  as a starting point.

THEOREM 3. Assume that:

- a) The linear program (1) is solvable.
- b) The gradients of the active constraints are linearly independent at the optimal point  $\bar{x} = p_2(0, \bar{X})$ .
- c) The strict complementarity holds for the linear program (1) at the optimal point  $\bar{x} = p_2(0, \bar{X})$ .

Let  $\{u^k, v^k\}$  be the sequence produced by the algorithm and let

$$I^k := \{i : u_i^k > 0\}, \quad J^k := \{j : v_j^k > 0\}, \quad B^k := \begin{bmatrix} E \\ J^k \\ \text{---} \\ A \\ I^k \end{bmatrix}. \quad (24)$$

Then there exist a  $\bar{k}$  such that for all  $k > \bar{k}$  it follows that the matrix  $B^k$  has full rank and

$$x^k := (B^k)^T s^k = \bar{x} = p_2(0, \bar{X})$$

where



$$s^k := [B^k(B^k)^T]^{-1} \begin{bmatrix} 0 \\ |J^k| \\ b \\ I^k \end{bmatrix}. \quad (25)$$

PROOF. By using Proposition 3 we have that there exist a  $\varepsilon^{**}$  and a  $k^*$  such that  $\forall \varepsilon \in (0, \varepsilon^{**})$  and  $\forall k > k^*$  we have (by recalling the definitions given in section 2)

$$I^k = I_a(\bar{x}) = I_b(\bar{x}), \quad J^k = J_a(\bar{x}) = J_b(\bar{x})$$

and, hence

$$B^k = B^k(\bar{x}), \quad s^k = s(\bar{x}).$$

Therefore, there exists a  $\bar{k}$  large enough such that  $\varepsilon^{\bar{k}} < \varepsilon^{**}$  and  $N^{\bar{k}} > k^*$  and, hence, the proof of theorem follows from part a) of Proposition 1. ■

REMARK 3. Practically we do not need to compute the matrix  $(B^k(B^k)^T)^{-1}$ , we only have to solve the system

$$(B^k(B^k)^T)s^k = \begin{bmatrix} 0 \\ |J^k| \\ b \\ I^k \end{bmatrix}.$$

Therefore any efficient sparsity preserving method for solving linear systems of equations can be used.

THEOREM 4. Assume that:

- a) The linear program (1) has a unique solution  $\bar{x}$ .
- b) The gradients of the active constraints are linearly independent at the optimal point  $\bar{x}$ .

Let  $\{u^k, v^k\}$  be the sequence produced by the algorithm and let

$$\begin{aligned}
 I^k &= \{i : u_i^k > 0\}, \quad J^k := \{i : v_i^k > 0\} \\
 \tilde{I}^k &= \{i : u_i^k = 0\}, \quad \tilde{J}^k := \{i : v_i^k = 0\} .
 \end{aligned}
 \tag{26}$$

Then there exist a  $k$  such that for all  $k > \bar{k}$  the matrix  $A_{I^k J^k}$  is a square non singular matrix and

$$x^k := (A_{I^k J^k})^{-1} b_{I^k} = \bar{x} .$$

PROOF. By the assumptions a) and b) we have that the linear program (4) and its dual have unique optimal solutions and that the strict complementarity holds (see page 26 of [11]). Now, by using again Proposition 3 we have that there exist a  $\epsilon^{**}$  and  $k^*$  such that  $\forall \epsilon \in (0, \epsilon^{**}]$  and  $\forall k > k^*$  we have

$$I^k = I_a(\bar{x}) = I_b(\bar{x}) = I_d(\bar{u}), \quad J^k = J_a(\bar{x}) = J_b(\bar{x}) = J_d(\bar{u})$$

$$\tilde{I}^k = I_n(\bar{x}) = \{i : \bar{u}_i = 0\}, \quad \tilde{J}^k = J_n(\bar{x}) = \{i : \bar{v}_i = 0\}$$

where  $(\bar{u}, \bar{v})$  is the optimal solution of the dual of the linear program (1).

Therefore for values of  $k$  large enough (such that  $\epsilon^k < \epsilon^{**}$  and  $k > k^*$ ) the proof of theorem follows by using usual arguments of the duality theory for linear programs (see [11], page 45).

■

In Algorithm I different stopping criterions can be used. As examples we propose two algorithms. In Algorithm II the stopping criterion is based on Theorem 2 and Theorem 3 and it uses the properties of the multiplier function  $\lambda(x)$  for the linear program (1) (we refer to [9] and [10] for the definition and the properties of the multiplier function  $\lambda(x)$ , here we recall only that  $\lambda(\cdot)$  is a function from  $R^n$  to  $R^{m+n}$  such that if  $\bar{x}$  is optimal for the linear program (1)  $\lambda(\bar{x})$  gives the optimal solution of the dual of the linear program).

Algorithm III is based on Theorem 4 and the duality theory for the linear programs.

Again Algorithm II and Algorithm III make use of the preselected sequences  $\{\epsilon^k\}$  and  $\{N^k\}$  such that (23) holds and the sequence  $\{I^k\}$ ,  $\{J^k\}$ ,  $\{\tilde{I}^k\}$ ,  $\{\tilde{J}^k\}$ ,  $\{B^k\}$  and  $\{s^k\}$  given by (24), (25) and (26).

#### ALGORITHM II

Data:  $(u^0, v^0) \in R_+^{m+n}$

Step 0: Set  $k = 1$ .

Step 1: Compute  $(u^k, v^k)$  by applying  $N^k$  iterations of the SOR algorithm to the problem (12) with  $\epsilon = \epsilon^k$  and by using  $(u^{k-1}, v^{k-1})$  as starting point.

Step 2: If  $B^k$  has full rank,  $u_{I^k}^k > -\epsilon^k s_{I^k}^k$ ,  $v_{J^k}^k > \epsilon^k s_{J^k}^k$ ,  $[E - B^k(B^k(B^k)^T)^{-1}B^k]c = 0$  go to step 3; else go to step 5.

Step 3: Set  $x^k = (B^k)^T s^k$  and  $\lambda^k = \lambda(x^k)$ .

Step 4: If  $(x^k, \lambda^k)$  satisfy the KKT conditions for the linear program (1) stop; else go to step 5.

Step 5: Set  $k = k+1$ , go to step 1.

REMARK 4. Step 2 is not necessary, it serves only to reduce the numbers of computations of the multiplier function. If the computation of the matrix  $(B^k(B^k)^T)^{-1}$  is too expensive we can replace step 2 with

Step 2': If  $K = iM$ , with  $i = 1, 2, \dots$ , go to step 3; else go to step 5.

where  $M$  is a fixed positive number.

COROLLARY 1. Under the assumption a), b) and c) of Theorem 3, Algorithm II terminates in a finite number of iterations and the produced point  $x^v$  is the

least 2-norm solution of the linear program (1) and  $\lambda^v$  is the optimal solution of its dual.

PROOF. The proof follows by repeating similar arguments of proof of Theorem 3 and by using part c) of Theorem 2 and the properties of the multiplier function.

#### ALGORITHM III

Data:  $(u^0, v^0) \in R_+^{m+n}$ .

Step 0: Set  $k = 1$ .

Step 1: Compute  $(u^k, v^k)$  by applying  $N^k$  iteration of the SOR algorithm to the problem (12) with  $\varepsilon = \varepsilon^k$  and by using  $(u^{k-1}, v^{k-1})$  as starting point.

Step 2: If  $A_{I^k \tilde{J}^k}$  is a square nonsingular matrix go to step 3; else go the step 5;

Step 3: Set  $z^k = (A_{I^k \tilde{J}^k})^{-1} b_{I^k}$ ,  $w^k = (A_{I^k \tilde{J}^k}^T)^{-1} c_{\tilde{J}^k}$ .

Step 4: If  $z^k > 0$ ,  $(A_{\tilde{I}^k \tilde{J}^k} z^k - b_{\tilde{I}^k}) < 0$ ,  $w^k > 0$ ,  $(A_{I^k J^k}^T w^k - c_{J^k}) > 0$  stop; else go to step 5.

Step 5: Set  $k = k+1$ , go to step 1.

COROLLARY 2. Under the assumption a) and b) of Theorem 4, Algorithm III terminates in a finite number of iterations and if  $(z^v, w^v)$  is the produced point and if  $(x^v, u^v)$  is defined as follows

$$x_1^v = \begin{cases} z_1^v & \text{if } i \in \tilde{J}^k \\ 0 & \text{if } i \in J^k \end{cases}, \quad u_1^v = \begin{cases} w_1^v & \text{if } i \in I^k \\ 0 & \text{if } i \in \tilde{I}^k \end{cases}.$$

Then  $x^v$  is the unique optimal solution of the linear program (1) and  $u^v$  is the optimal solution of its dual.

PROOF. The proof follows from Theorem 4 and from the duality theory for linear programs (see [11], pages 45-50).

For brevity's sake, here we do not discuss an interesting algorithm which is the conjunction of Algorithm I of this paper with the algorithm proposed in [6]. In fact such an algorithm would be linearly or superlinearly convergent under assumptions weaker than those of Theorem 3 and would also have finite termination under the assumption a) - c) of Theorem 3.

#### ACKNOWLEDGEMENT

The author wishes to thank Professor O. L. Mangasarian for helpful discussions and comments during the preparation of this paper. The author is also indebted to Dr. R. De Leone for pointing out Theorem 2.8 of [11].

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4. TITLE (and Subtitle) A FINITE ALGORITHM FOR THE LEAST TWO-NORM SOLUTION OF A LINEAR PROGRAM		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Stefano Lucidi		8. CONTRACT OR GRANT NUMBER(s) DCR-8420963 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53705		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 5 - Optimization and Large Scale Systems
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE July 1986
		13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709 National Science Foundation Washington, DC 20550		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Linear programming, Quadratic programming		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) By perturbing properly a linear program to a separable quadratic program it is possible to solve the latter in its dual variable space by iterative techniques such as sparsity-preserving SOR (successive overrelaxations). In this way large sparse linear programs can be handled.  In this paper we give a new computational criterion to check whether the solution of the perturbed quadratic programs provide the least 2-norm solution of the original linear program. This criterion improves on the criterion		



ABSTRACT (continued)

proposed in an earlier paper.

We also describe an algorithm for solving linear programs which is based on the SOR methods. The main property of this algorithm is that, under mild assumptions, it finds the least 2-norm solution of a linear program in a finite number of iterations.

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